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Distance-two labelings of digraphs

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Abstract

For positive integers $j \geq k$, an $L(j,k)$-labeling of a digraph D is a function f from $V(D)$ into the set of nonnegative integers such that $|f(x)-f(y)| \geq j$ if x is adjacent to y in D and $|f(x)-f(y)| \geq k$ if x is of distant two to y in D. Elements of the image of f are called labels. The $L(j,k)$-labeling problem is to determine the $\vec{\lambda}_{j,k}$-number $\vec{\lambda}_{j,k}(D)$ of a digraph D, which is the minimum of the maximum label used in an $L(j,k)$-labeling of D. This paper studies $\vec{\lambda}_{j,k}$-numbers of digraphs. In particular, we determine $\vec{\lambda}_{j,k}$-numbers of digraphs whose longest dipath is of length at most 2, and $\vec{\lambda}_{j,k}$-numbers of ditrees having dipaths of length 4. We also give bounds for $\vec{\lambda}_{j,k}$-numbers of bipartite digraphs whose longest dipath is of length 3. Finally, we present a linear-time algorithm for determining $\vec{\lambda}_{j,1}$-numbers of ditrees whose longest dipath is of length 3.

Keywords. $L(j,k)$-labeling, digraph, ditree, homomorphism, algorithm.

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1 Introduction

For positive integers $j \geq k$, an $L(j,k)$-*labeling* of a graph G is a function f from $V(G)$ into the set of nonnegative integers such that $|f(x) - f(y)| \geq j$ if x is adjacent to y in G and $|f(x)-f(y)| \geq k$ if x is of distance two to y in G. Elements of the image of f are called *labels*, and the *span* of f is the difference between the largest and the smallest labels of f. The minimum span taken over all $L(j,k)$-labelings of G, denoted by $\lambda_{j,k}(G)$, is called the $\lambda_{j,k}$-*number* of G. And, if f is a labeling with a minimum span, then f is called a $\lambda_{j,k}$-*labeling* of G. We shall assume without loss of generality that the minimum label of an $L(j,k)$-labeling of G is 0. We use λ_j for $\lambda_{j,1}$ and λ for $\lambda_{2,1}$ for short.

A variation of Hale's channel assignment problem [16], the problem of labeling a graph with a condition at distance two, was first investigated in the case of $j = 2$ and $k = 1$ by Griggs and Yeh [15]. They derived formulas for the λ-numbers of paths and cycles and established bounds on the λ-numbers of trees and n-cubes. They also investigated the relationship between $\lambda(G)$ and other graph invariants such as $\chi(G)$ and $\Delta(G)$. Other authors have subsequently contributed to the literature of $L(j,k)$-labelings with focus on the case of $j = 2$ and $k = 1$, see the references.

In this paper we consider the channel assignments in which frequency inference has direction. The formulation then becomes $L(j,k)$-labelings on digraphs rather than on graphs. Recall that in a digraph D, the *distance* $d_D(x,y)$ from vertex x to vertex y is the length of a shortest dipath from x to y. We then may define $L(j,k)$-labelings for digraphs in precisely the same way as for graphs. However, to distinguish from the notation for graphs, we use $\vec{\lambda}_{j,k}$-number $\vec{\lambda}_{j,k}(D)$ for a digraph D. We also use $\vec{\lambda}_j$ for $\vec{\lambda}_{j,1}$ and $\vec{\lambda}$ for $\vec{\lambda}_{2,1}$.

The $\vec{\lambda}_{1,1}$-number of a digraph is closed related to its oriented chromatic number. For a digraph D, an *oriented labeling* is a function f from $V(D)$ into the set of positive integers such that $f(x) \neq f(y)$ for $xy \in E(D)$ and whenever an ordered pair (p,q) is used for an edge xy as $(f(x), f(y))$, the ordered pair (q,p) is never used for any other edge. The *oriented chromatic number* $\vec{\chi}(D)$ of a digraph D is the minimum size of the image of an oriented labeling of D. Oriented chromatic numbers have been studied in the literature extensively. Notice that $\vec{\lambda}_{1,1}(D) \leq \vec{\chi}(D) - 1$ for any digraph D.

For a tree T, Griggs and Yeh [15] showed that $\Delta(T)+1 \le \lambda(T) \le \Delta(T)+2$; and a polynomial-time algorithm to determine the value of $\lambda(T)$ (respectively, $\lambda_j(T)$) was given by Chang and Kuo [3] (respectively, Chang et al. [2]). A surprising result by Chang and Liaw [5] says that $\vec{\lambda}(T) \le 4$ for any ditree T, which is an orientation of a tree. Suppose the largest length of a dipath in the ditree T is ℓ. They also proved that $\vec{\lambda}(T)=2$ if $\ell=1$; $\vec{\lambda}(T)=3$ if $\ell=2$; $3 \le \vec{\lambda}(T) \le 4$ if $\ell=3$; and $\vec{\lambda}(T)=4$ if $\ell \ge 4$. Determining the exact value of $\vec{\lambda}(T)$ for the case of $\ell=3$ left open, while there are examples showing that $\vec{\lambda}(T)$ can be 3 or 4.

The main results of this paper is to determine the exact value of $\vec{\lambda}_{j,k}(D)$ for a digraph D whose longest dipath is of length 1 or 2. It is also proved that $j+k \le \vec{\lambda}_{j,k} \le j+2k$ for a bipartite digraph whose longest dipath is of length 3. Finally, a linear-time algorithm is given for determining $\vec{\lambda}_j(T)$ of a ditree T whose longest dipath is of length 3.

2 Preliminary

In this section, we first fix some notation and terminology, and then derive some general propositions for $\lambda_{j,k}$-numbers of digraphs.

For a graph G, if D is the digraph resulting from G by replacing each edge $\{x,y\}$ by two (directed) edges xy and yx, then $\vec{\lambda}_{j,k}(D)=\lambda_{j,k}(G)$. However, *in this paper all digraphs are assumed to be strongly simple*, i.e. it has no loops or multiple edges or both the edges xy and yx.

A digraph D is *homomorphic* to another digraph H if there is a *homomorphism* from D to H, which is a function $h: V(D) \to V(H)$ such that $xy \in E(D)$ implies $h(x)h(y) \in E(H)$. Define

$$N_D^+(v)=\{u : vu \in E(D)\}, \quad N_D^-(v)=\{u : uv \in E(D)\}, \quad N_D(v)=N_D^+(v) \cup N_D^-(v).$$

If there is no confusion on the digraph D, we simply use $N^+(v)$ for $N_D^+(v)$, $N^-(v)$ for $N_D^-(v)$ and $N(v)$ for $N_D(v)$. We call the vertices in $N^+(v)$ the *out-neighbors* of v, in $N^-(v)$ the *in-neighbors* and in $N(v)$ the *neighbors*. A *source* is a vertex with no in-neighbors, and a *sink* a vertex with no out-neighbors. A *leaf* of a digraph is a vertex with exactly one neighbor.

An *orientation* of a graph is a digraph obtained from the graph by assigning each edge

of the graph an direction. The *underlying graph* of a digraph is the graph obtained from the digraph by forgetting the directions of its edges.

The n-*dipath* is the digraph $\vec{P}_n$ with $V(\vec{P}_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(\vec{P}_n) = \{v_0v_1, v_1v_2, \ldots, v_{n-2}v_{n-1}\}$. The n-*dicycle* is the digraph $\vec{C}_n$ with $V(\vec{C}_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(\vec{C}_n) = \{v_0v_1, v_1v_2, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_0\}$. The n-*path* P_n is the underlying graph of the n-dipath $\vec{P}_n$, and the n-*cycle* C_n is the underlying graph of the n-dicycle $\vec{C}_n$. A *ditree* is an orientation of a tree. Notice that a nontrivial ditree has at least two leaves. A digraph is *bipartite* if and only if its underlying graph is bipartite.

Lemma 1 *If D is a subdigraph of a digraph H, then $\vec{\lambda}_{j,k}(D) \le \vec{\lambda}_{j,k}(H)$.*

Proof. The lemma follows from the fact that the restriction of an $L(j,k)$-labeling of H on $V(D)$ is an $L(j,k)$-labeling of D, since $1 \le d_H(x,y) \le d_D(x,y)$ for any two distinct vertices x and y in D. ∎

Given n digraphs $D_1, D_2, \cdots, D_n$, the *union* of these n digraphs, denoted by $\bigcup_{i=1}^n D_i$, is the digraph D with $V(D) = \bigcup_{i=1}^n V(D_i)$ and $E(D) = \bigcup_{i=1}^n E(D_i)$. The following lemma is obvious.

Lemma 2 *If $D = \bigcup_{i=1}^n D_i$, then $\vec{\lambda}_{j,k}(D) = \max\limits_{1\le i\le n} \vec{\lambda}_{j,k}(D_i)$.*

Lemma 3 *If digraph D is homomorphic to digraph H, then $\vec{\lambda}_{j,k}(D) \le \vec{\lambda}_{j,k}(H)$.*

Proof. The lemma follows from the fact that the composition of a homomorphism h from D to H with an $L(j,k)$-labeling of H is an $L(j,k)$-labeling of D, since H being strongly simple implies that $1 \le d_H(h(x), h(y)) \le d_D(x,y)$ for any two vertices x and y in D with $1 \le d_D(x,y) \le 2$. ∎

Lemma 4 *The following statements hold for any digraph D.*

(1) $\vec{\lambda}_{j,k}(D) = 0$ *if and only if D has no edge.*

(2) *If D has at least one edge, then $\vec{\lambda}_{j,k}(D) \ge j$.*

(3) *For any digraph D with at least one edge, $\vec{\lambda}_{j,k}(D) = j$ if and only if every vertex is either a source or a sink.*

Proof. Statements (1) and (2) are trivial.

(3) By statement (2), we have $\vec{\lambda}_{j,k}(D) \geq j$. If every vertex of D is either a source or a sink, then consider the mapping f defined by $f(x) = 0$ if x is a source and $f(x) = j$ otherwise. It is easy to check that f is an $L(j,k)$-labeling of D, which implies that $\vec{\lambda}_{j,k}(D) \leq j$ and so $\vec{\lambda}_j(D) = j$. On the other hand, if D has a vertex y which is neither a source nor a sink, then choose $x \in N^-(y)$ and $z \in N^+(y)$. For any $L(j,k)$-labeling g, we have $g(x) \neq g(z)$ and $g(y)$ differs from $g(x)$ and $g(z)$ by at least j. These imply that g must use a label greater than j, i.e., $\vec{\lambda}_{j,k}(D) > j$. ∎

Notice that in general $\vec{\lambda}_{j,k}(\vec{P}_n) = \lambda_{j,k}(P_n)$ and $\vec{\lambda}_{j,k}(\vec{C}_n) = \lambda_{j,k}(C_n)$. For the purpose of this paper, we need the values [8]: $\vec{\lambda}_{j,k}(\vec{P}_3) = \vec{\lambda}_{j,k}(\vec{P}_4) = j+k$, $\vec{\lambda}_{j,k}(\vec{P}_5) = \min\{2j, j+2k\}$, $\vec{\lambda}_{j,k}(\vec{C}_3) = 2j$ and $\vec{\lambda}_{j,k}(\vec{C}_4) = j+2k$.

3 Digraphs with a specified longest dipath length

This section investigates digraphs in which the length ℓ of a longest dipath is at most 3.

The case when $\ell = 1$ is a consequence of Lemma 4 (3), as a longest dipath of a digraph is of length 1 if and only if the digraph has at least one edge and every vertex is either a source or a sink.

We now consider the case when $\ell = 2$. There are two subcases. We first deal with the case when D is a bipartite digraph.

Theorem 5 *For any bipartite digraph D whose longest dipath has length 2, we have $\vec{\lambda}_{j,k}(D) = j+k$*

Proof. According to Lemma 1, $\vec{\lambda}_{j,k}(D) \geq \vec{\lambda}_{j,k}(\vec{P}_3) = j+k$ since D has a dipath of length 2.

On the other hand, choose a bipartition $A \cup B$ of $V(D)$. Define function f on $V(D)$

by

$$f(x) = \begin{cases} 0, & \text{if } x \in A - S; \\ k, & \text{if } x \in A \cap S; \\ j, & \text{if } x \in B \cap S; \\ j+k, & \text{if } x \in B - S. \end{cases}$$

We shall check that f is an $L(j,k)$-labeling of D as follows, which gives $\vec{\lambda}_{j,k} \le j+k$.

If $d_D(x,y) = 1$, then x and y are in different parts, and y is not a source. In other words, either $x \in A$ with $y \in B - S$ or $x \in B$ with $y \in A - S$. Then, $|f(x) - f(y)| \ge j$.

If $d_D(x,y) = 2$, then there is a dipath x, w, y. First, x and y are in the same part, and y is not a source. Second, suppose x is not a source, i.e. x has an in-neighbor z. Since D is strongly simple, $z \ne w$; and since D is bipartite, $z \ne y$. These give a dipath z, x, w, y of length 3, which is impossible. So, x is a source. Therefore, either $x \in A \cap S$ with $y \in A - S$ or $x \in B \cap S$ with $y \in B - S$. In either case, $|f(x) - f(y)| \ge k$.

Thus, f is an $L(j,k)$-labeling of D and so $\vec{\lambda}_{j,k}(D) \le j+k$. ∎

We next consider the case when $\ell = 2$ and D is not bipartite.

Theorem 6 *For any non-bipartite digraph D whose longest dipath is of length 2, we have $\vec{\lambda}_{j,k}(D) = 2j$*

Proof. According to Lemma 2, we may assume that D is connected.

For the case when D is cyclic, since D is strongly simple, $D = \vec{C}_3$ for otherwise there is a dipath of length 3. In this case, $\vec{\lambda}_{j,k}(D) = \vec{\lambda}_{j,k}(\vec{C}_3) = 2j$.

Now, suppose D is acyclic. Let S_1 denote the set of all sources, S_2 all vertices which are neither a source nor a sink, and S_3 all sinks. Then $S_1 \cup S_2 \cup S_3$ is a partition of $V(D)$. Define function f on $V(D)$ by $f(x) = (p-1)j$ for $x \in S_p$. As $|f(x) - f(y)| \ge j$ for $x \in S_p$ and $y \in S_q$ with $p \ne q$, in order to check that f is an $L(j,k)$-labeling, we only need to show that any S_p can not contain two distinct vertices x and y of distance one or two. Suppose to the contrary that such S_p, x and y exist. It is then obvious that $p = 2$. By the definition of S_2, we may choose an in-neighbor w of x and an out-neighbor z of y. Since D is acyclic, wPz is a dipath of length at least 3, which is impossible. Therefore, f is an $L(j,k)$-labeling of D and so $\vec{\lambda}(D) \le 2j$.

On the other hand, suppose D has an $L(j,k)$-labeling f using labels in $\{0,1,\ldots,2j-1\}$. Choose an odd cycle (not necessarily directed) $(v_0,v_1,v_2,\ldots,v_{n-1})$ in D. For any p, one of $f(v_p)$ and $f(v_{pi+1})$ must be in $S=\{0,1,\ldots,j-1\}$ and the other in $B=\{j,j+1,\ldots,2j-1\}$. But this is impossible as n is odd. Therefore, $\vec{\lambda}_{j,k}(D)\geq 2j$. ∎

For the case when $\ell=3$, we only consider bipartite digraphs.

Theorem 7 *For any bipartite digraph D whose longest dipath has length 3, we have $j+k\leq \vec{\lambda}_{j,k}(D)\leq j+2k$*

Proof. According to Lemma 1, $\vec{\lambda}_{j,k}(D)\geq\vec{\lambda}_{j,k}(\vec{P}_4)=j+2k$ since D has a dipath of length 3.

On the other hand, according to Lemma 2, we may assume that D is connected. If D contains a $\vec{C}_4$, then $D=\vec{C}_4$ for otherwise there is a dipath of length 4. Therefor, $\vec{\lambda}_{j,k}(D)=\vec{\lambda}_{j,k}(\vec{C}_4)=j+2k$. Now, we may that $\vec{C}_4\nsubseteq D$. Denote by S the set of all sources and all vertices whose in-neighbors are all sources. Choose a bipartition $A\cup B$ of $V(D)$. Then, define function f on $V(D)$ by

$$f(x) \;=\; \begin{cases} 0, & \text{if } x\in A-S;\\ k, & \text{if } x\in A\cap S;\\ j+k, & \text{if } x\in B\cap S;\\ j+2k, & \text{if } x\in B-S.\end{cases}$$

We check that f is an $L(j,k)$-labeling of D as follows. If $|f(x)-f(y)|<j$ for some x adjacent to y, then x and y are in a same part, which contradicts that D is bipartite. If $|f(x)-f(y)|<k$ for some x and y with $d_D(x,y)=2$, then x and y are both in one of the sets $A-S$, $A\cap S$, $B\cap S$ and $B-S$. By the definition of S, condition $d_D(x,y)=2$ implies $y\notin S$. This in turn implies that $x\notin S$. Again, by the definition of S, there is a vertex z whose distance to x is 2. Since, D is strongly simple, bipartite and contains no $\vec{C}_4$, the vertices z,x,y then creates a dipath of length 4, a contradiction. Thus, f is an $L(j,k)$-labeling of D. These prove the theorem. ∎

4 Ditrees

In this section, we studies $\vec{\lambda}_{j,k}$-numbers for ditrees. According to the results in the previous section, we only consider the case when the longest dipath is of length at least 3.

First, a useful lemma.

Lemma 8 *If T is a ditree and $n \geq 3$, then there is a homomorphism from T to $\vec{C}_n$.*

Proof. The lemma is trivial when T has exactly one vertex. Suppose T has at least two vertices. Choose a leaf x with (necessarily) exactly one neighbor y. By the induction hypothesis, there is a homomorphism h from $T-x$ to $\vec{C}_n$. Suppose $h(y) = v_i$. We may extend h to a homomorphism from T to $\vec{C}_n$ by letting $h(x) = v_{i+1}$ if $x \in N^+(y)$ and $h(x) = v_{i-1}$ if $x \in N^-(y)$, where the addition/substraction in $i+1$ or $i-1$ are taken modula n. The lemma then follows from induction. ∎

Theorem 9 *For any ditree T, we have $\vec{\lambda}_{j,k}(T) \leq \min\{2j, j+2k\}$. Moreover, $\vec{\lambda}_{j,k}(T) = \min\{2j, j+2k\}$ if T has a dipath of length 4.*

Proof. According to Lemma 8, there is a homomorphism from T to $\vec{C}_3$ (respectively, $\vec{C}_4$). By Lemma 3, $\vec{\lambda}_{j,k}(T) \leq \vec{\lambda}_{j,k}(\vec{C}_3) = 2j$ (respectively, $\vec{\lambda}_{j,k}(T) \leq \vec{\lambda}_{j,k}(\vec{C}_4) = j+2k$). On the other hand, suppose T has a dipath of length 4. By Lemma 1, $\vec{\lambda}_{j,k}(T) \geq \vec{\lambda}_{j,k}(\vec{P}_5) = \min\{2j, j+2k\}$ and so $\vec{\lambda}_j(T) = \min\{2j, j+2k\}$. ∎

So far, we have determined $\vec{\lambda}_{j,k}$-numbers for all ditrees except for the case when its longest dipath is of length 3. In the rest of this paper, we shall give an algorithm to determine the value of $\vec{\lambda}_j(T)$ for a ditree T whose longest dipath is of length 3. In this case, according to Theorem 7 either $\vec{\lambda}_j(T) = j+1$ or $j+2$. Below are two examples showing that the two possibilities happen. Consider the ditree $T_1 = \vec{P}_4$ and the ditree $T_2 = (V_2, E_2)$ with

$$V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \quad \text{and} \quad E_2 = \{v_1v_2, v_2v_3, v_3v_4, v_5v_4, v_5v_6, v_6v_7, v_7v_8\}.$$

It is the case that the longest dipaths of both T_1 and T_2 are of length 3, but $\vec{\lambda}_j(T_1) = j+1$ while $\vec{\lambda}_j(T_2) = j+2$ when $j \geq 2$.

In sprite of the algorithm for λ-nmbers of trees [3], we now give an algorithm to determine if a ditree has an $L(j,1)$-labeling of span $j+1$, without any assumption on length of a longest dipath in it. For this purpose, we consider T as rooted at a vertex v and denote the ditree as T_v if necessary. Let T_v^+ (respectively, T_v^-) denote the ditree obtained from T_v

by adding a new vertex v^+ (respectively, v^-) and a new edge vv^+ (respectively, v^-v). We consider T_v^+ as rooted at v^+ and T_v^- rooted at v^-. Denote

$$S(T_v^+) = \{(a,b) : a = f(v^+) \text{ and } b = f(v) \text{ for some } L(j,1)\text{-labeling } f \text{ of span } j+1 \text{ for } T_v^+\},$$

$$S(T_v^-) = \{(a,b) : a = f(v^-) \text{ and } b = f(v) \text{ for some } L(j,1)\text{-labeling } f \text{ of span } j+1 \text{ for } T_v^-\}.$$

Note that $\vec{\lambda}_j(T_v^+) \leq j+1$ if and only if $S(T_v^+) \neq \emptyset$, and $\vec{\lambda}_j(T_v^-) \leq j+1$ if and only if $S(T_v^-) \neq \emptyset$. Suppose T' is a ditree for which we want to determine if $\vec{\lambda}_j(T') \leq j+1$. Choose any leaf y in T' who only neighbor is v and let T be the ditree obtained from T' by deleting y and the edge incident with y. If we view T as rooted at v, then T' is equal to either T_v^+ (when vy is an edge in T') or T_v^- (when yv is an edge in T').

We note that $S(T_v^+)$ and $S(T_v^-)$ are subsets of the set

$$W = \{(0,j),(0,j+1),(1,j+1),(j,0),(j+1,0),(j+1,1)\}.$$

For any $a \in \{0,1,j,j+1\}$ and $S \subseteq W$, let $S_a = \{b : (a,b) \in S\}$. Notice that S_a is of size at most 2. In fact $S_a \subseteq W_a$ and

$$W_0 = \{j, j+1\}, W_1 = \{j+1\}, W_j = \{0\}, W_{j+1} = \{0,1\}.$$

For any $(a,b) \in W$ and $S \subseteq W$, let $S_{(a,b)} = \{b' : (a,b') \in S - \{(a,b)\}\}$. Notice that $S_{(a,b)}$ is of size at most 1. In fact $S_{(a,b)} \subseteq W_{(a,b)}$ and

$$W_{(0,j)} = \{j+1\}, W_{(0,j+1)} = \{j\}, W_{(1,j+1)} = W_{(j,0)} = \emptyset, W_{(j+1,0)} = \{1\}, W_{(j+1,1)} = \{0\}.$$

Suppose $T_v - v$ contains $r+s$ ditrees $T_{u_1}, T_{u_2}, \ldots, T_{u_r}, T_{v_1}, T_{v_2}, \ldots, T_{v_s}$, where each u_i is *adjacent to* v and each v_i is *adjacent from* v in T_v. Note that T_v can be considered as identifying $u_1^+, u_2^+, \ldots, u_r^+, v_1^-, v_2^-, \ldots, v_s^-$ to a vertex v on the disjoint union of $T_{u_1}^+, T_{u_2}^+, \ldots, T_{u_r}^+, T_{v_1}^-, T_{v_2}^-, \ldots, T_{v_s}^-$. We then have

Theorem 10 *For any ditree T, we have $S(T_v^+) =$*

$$\{(a,b) \in W : S(T_{u_p}^+)_{(b,a)} \neq \emptyset \textit{ for } 1 \leq p \leq r; \emptyset \neq S(T_{v_q}^-)_b \neq S(T_{u_1}^+)_{(b,a)} \textit{ for } 1 \leq q \leq s\},$$

where $S(T_{u_1}^+)_{(b,a)}$ is assume to be $\emptyset$ if $r = 0$. Also, $S(T_v^-) =$

$$\{(a,b) \in W : S(T_{v_q}^-)_{(b,a)} \neq \emptyset \textit{ for } 1 \leq q \leq s; \emptyset \neq S(T_{u_i}^-)_b \neq (T_{v_1}^+)_{(b,a)} \textit{ for } 1 \leq p \leq r\},$$

where $S(T_{v_1}^+)_{(b,a)}$ is assume to be $\emptyset$ if $s = 0$.

Proof. The first equality follows from that fact that T_v^+ has an $L(j,1)$-labeling f with $f(v^+) = a$ and $f(v) = b$ if and only if $T_{u_p}^+$ has an $L(j,1)$-labeling g_p with $g_p(u_p^+) = b$ but $g_i(u_p) \neq a$ for $1 \leq p \leq r$, and $T_{v_q}^-$ has an $L(j,1)$-labeling h_q with $h_q(v_q^-) = b$ but $h_q(v_q) \neq g_1(u_1)$ for $1 \leq q \leq s$.

The second equality follows from a similar argument. ∎

We remark that in the calculation of $S(T_v^+)$, the condition "$S(T_{v_q}^-)_b \neq S(T_{u_1}^+)_{(b,a)}$ for $1 \leq q \leq s$" is redundant if $r = 0$. When $r \geq 1$ and $S(T_{u_p}^+)_{(b,a)} \neq \emptyset$ for $1 \leq q \leq s$, it is the case that these sets are equal to a same set. Similar statements are true for the set $S(T_v^-)$.

Having the theorem above, we then can compute the sets $S(T_v^+)$ and $S(T_v^-)$ recursively, using the initial conditions that $S(T_v^+) = S(T_v^-) = W$ when T is a ditree of just one vertex. This gives a linear-time algorithm to determine whether $\vec{\lambda}_j(T) = j+1$ or not for a ditree T, without any assumption on the length of the longest dipath in T.

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